PENRITH HIGH SCHOOL

2013
HSC TRIAL EXAMINATION

Mathematics Extension 1

General Instructions:

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In questions 11 – 14, show relevant mathematical reasoning and/or calculations

Total marks–70

SECTION I **Pages 3–5**

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

SECTION II **Pages 6–9**

60 marks

- Attempt Questions 11–14
- Allow about 1 hours 45 minutes for this section

Student Name: ____________________ **Teacher Name:** ____________________

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Student Name: ____________________ **Teacher Name:** ____________________

Exam Outcomes

1. Use the relationship between functions, inverse functions and derivatives (**Differential Calculus**)
2. Study of simple harmonic and projectile motion (**Motion**)
3. Manipulate polynomial functions (**Polynomials**)
4. Applies angle and chord properties of the circle (**Circle Geometry**)
5. Problem solving (**PS**)

Paper Grid

Outcome	Mark	Qn-Num/Out-of	Mark
Differential Calculus	/33	1 / 1 4 / 1 6 / 1 8 / 1 9 / 1 10 /1	/6
		11-a / 1 11-e / 3	/4
		12-a / 1 12-b / 5 12-c / 2 12-e / 4	/12
		13-a / 2 13-d / 7	/9
		14-b / 2	/2
Motion	/13	14 -a / 3 14-c / 10	/13
Polynomials	/5	2 / 1 7 / 1	/2
		13-b / 3	/3
Circle Geometry	/3	12-d / 3	/3
Problem Solving	/16	3 / 1 5 / 1	/2
		11-b / 2 11-c / 2 11-d / 2 11 -f / 5	/11
		13-c / 3	/3
Exam Total	/70		%

Assessor: Daniel Antone

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Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple–choice answer sheet for Questions 1–10

1)! A point P moves in the xy-plane such that $P(\tan\theta, \cot\theta)$ is its parametric presentation with the parameter θ, where θ is any real number. The locus of P then is

(A) Parabola

(B) Circle

(C) Hyperbola

(D) Straight Line

2)! Let $P(x)$ be a polynomial of degree $n > 0$. Let $Q(x)$ be a polynomial of degree $m \le n$ such that

$$P(x) = (x - a)^r\, Q(x) + R(x)$$

Then the degree of $R(x)$ is

(A) $n + m + r$

(B) $n - m - r$

(C) $n + m - r$

(D) $n - m + r$

3)! The sum of this infinite geometric series $\sqrt{2} - 1 + \frac{1}{\sqrt{2}} - \frac{1}{2} + \frac{1}{2\sqrt{2}} - \ldots$ is closest to

(A) 0.5

(B) 1

(C) 1.5

(D) 2

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4)! Let $T(x)$ be a function defined by $T(x) = [f(x)g(x)]^{n+1}$, where $f(x)$ and $g(x)$ are two real functions. Then $\frac{dT}{dx}$ is

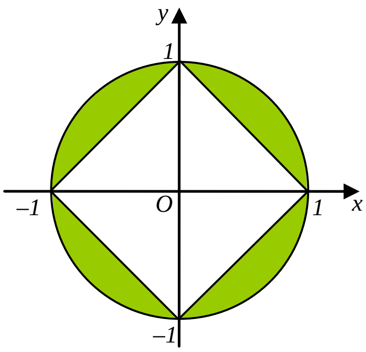
(A) $\frac{dT}{dx} = (n+1)[f(x)g(x)]^n \frac{df}{dx} \cdot \frac{dg}{dx}$

(B) $\frac{dT}{dx} = (n+1)[f(x)g(x)]^n [\frac{df}{dx} + \frac{dg}{dx}]$

(C) $\frac{dT}{dx} = (n+1)[f(x)g(x)]^n [f\frac{df}{dx} + g\frac{dg}{dx}]$

(D) $\frac{dT}{dx} = (n+1)[f(x)g(x)]^n [g\frac{df}{dx} + f\frac{dg}{dx}]$

5)! The only set of inequalities that represents the shaded regions between the circle and the square below is



(A) $x^2 + y^2 \le 1$ and $|x| + |y| \ge 1$

(B) $x^2 + y^2 \le 1$ and $|x| - |y| \ge 1$

(C) $x^2 + y^2 \le 1$ and $|x - y| \ge 1$

(D) $x^2 + y^2 \le 1$ and $|x + y| \le 1$

6)! Consider the functions $f(x) = e^x$ and $g(x) = \ln x$. Let a be a real number such that $a > 1$.

The only correct statement of the following is

(A) $f'(a) \le g'(a)$

(B) $f'(a) \ge g'(a)$

(C) $f'(a) < g'(a)$

(D) $f'(a) > g'(a)$

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7)! Let $f(x)$ be the cubic polynomial defined by $f(x) = (x - 1)^3 + x$. The point $(1, 1)$ is

(A) a stationary point of $f(x)$

(B) a turning point of $f(x)$

(C) a horizontal point of inflexion of $f(x)$

(D) a non-horizontal point of inflexion of $f(x)$

8)! The only correct statement about the function $f(\theta) = \dfrac{2\sin(\theta + 45)}{5\cos(45 - \theta)}$ is that

(A) it is a constant function

(B) it varies as θ varies

(C) it has a maximum value $0{\cdot}4$

(D) it has a minimum value $0{\cdot}4$

9)! The domain and range for the function $y = 2\cos^{-1}(x)$ is

(A) Domain: $-1 \le x \le 1$, Range: $0 \le y \le 2\pi$

(B) Domain: $-1 \le x \le 1$, Range: $0 \le y \le \pi$

(C) Domain: $0 \le x \le 1$, Range: $0 \le y \le 2\pi$

(D) Domain: $0 \le x \le 1$, Range: $0 \le y \le \pi$

10)! Consider $f(x) = \ln(x) - \ln(-x)$. Then $f(x)$ is

(A) An even function

(B) An odd function

(C) Undefined everywhere

(D) A relation which is not a function

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Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations

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Question 11 (15 marks) Start a NEW page

(a)! Find all exact values of the angle $tan^{-1}\left(-\sqrt{3}\right)$ in radians **1**

(b)! $A(-5, 6)$ and $B(1, 3)$ are two points. Find the coordinates of the point P which divides the interval AB externally in the ratio $5:2$. **2**

(c)! The perpendicular distance from the point (x_1, y_1) to the line $y = x + 3$ is $2\sqrt{2}$ and to x-axis is 3. Find the coordinates of the point (x_1, y_1). **2**

(d)! Find all possible solutions for the equation $(2x^2 - 1)^2 = \dfrac{(2x^2 - 1)^2}{(2 - 4x^2)}$. **2**

(e)! Show that the first derivative of the function $2x\sqrt{\sin x}$ may be given as $\dfrac{\cos x + 2\sin x}{\sqrt{\sin x}}$. **3**

(f)! Consider the following functions

$$f(x) = x^2 + 4x - 12 \text{ and } g(x) = \frac{x+6}{2-x}$$

(i) Solve the identity $f(x) \le 0$ and indicate your solution on a number line. **2**

(ii) Solve the identity $g(x) \le 0$ and indicate your solution on a number line. **2**

(iii) Find the simultaneous solution of the two inequalities $f(x) \le 0$ and $g(x) \le 0$. **1**

Proceed to next page for question (12)

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Question 12 (15 marks) Start a NEW page

(a)! A circular plate is being expanded by heating. When the radius just reaches a value of 20 cm, it (the radius) is increasing at the rate of $0{\cdot}01$ cm/s. Find the rate of increase in the area at this moment in terms of π. **1**

(b)! Consider the two exponential functions $y = e^{2x}$ and $y = e^x + 2$.

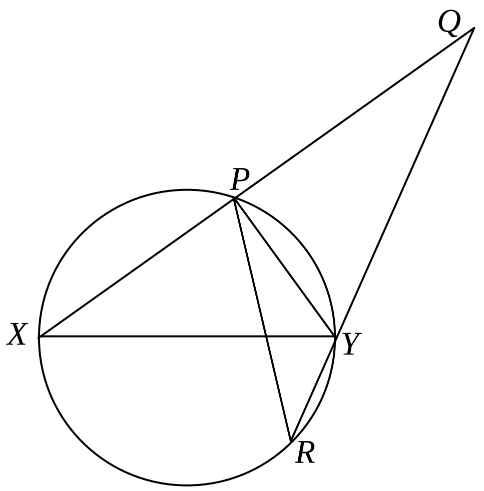
(i) Draw a neat sketch showing the graphs of $y = e^{2x}$ and $y = e^x + 2$ on the same diagram, showing any asymptotes and axes intercepts. **2**

(ii) Show that the coordinates of the point of intersection of $y = e^{2x}$ and $y = e^x + 2$ is $(ln2, 4)$. **1**

(iii) Find the area bounded by the y-axis and the two curves $y = e^{2x}$ and $y = e^x + 2$. Give your final answer correct to 2 decimal places. **2**

(c)! The size of the acute angle between the tangents drawn to the curve $y = lnx$ at the points where $x = 1$ and $x = x_1$ is $\dfrac{\pi}{6}$. Find the exact value of x_1. **2**

(d)! **3**



XY is a diameter in the circle above. Given that $\angle X = 35°$ and $\angle Q = 25°$, find the size of $\angle YPR$, giving reasons.

(e)! Let $y = sin^{-1}(1 - x^2)$.

(i) By using the substitution $u = 1 - x^2$, or otherwise, show that $\dfrac{dy}{dx} = \dfrac{-2}{\sqrt{1-x^2}}$. **1**

(ii) Hence show that $f'(x) = 0$ where $f(x) = 2\,cos^{-1}(\dfrac{x}{\sqrt{2}}) - sin^{-1}(1 - x^2)$. **1**

(iii) Hence or otherwise, show that $2\,cos^{-1}(\dfrac{x}{\sqrt{2}}) - sin^{-1}(1 - x^2) = \dfrac{\pi}{2}$. **2**

Proceed to next page for question (13)

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Question 13 (15 marks) Start a NEW page

(a)! Evaluate $\int_0^1 \sqrt{1-x^2}\,dx$ using the substitution $x = \sin\theta$. **2**

(b)! (i) State the conditions that the quadratic expression $ax^2 + bx + c$ is negative definite. **1**

(ii) Show that the expression $(k^2 + k)x^2 - (2k - 6)x + 2$, where $k \neq 0$, can never be negative definite. **1**

(iii) Find the range of values of k for which the expression is positive definite. **1**

(c)! Prove by mathematical induction that **3**

$2\times 1! + 5\times 2! + 10\times 3! + ... + (n^2+1)\times n! = n\times(n+1)!$ for all integers $n = 1, 2, 3, \ldots$

(d)! A machine which initially costs $\$49\ 000$ loses value at a rate proportional to the difference between its current value $\$M$ and its final scrap value $\$1000$. After 2 years the value of the machine is $\$25\ 000$.

(i) Explain why $\frac{dM}{dt} = -k(M - 1000)$ for some constant $k > 0$, and verify that $M = 1000 + Ae^{-kt}$, A constant, is a solution of this equation. **2**

(ii) Find the exact values of A and k. **3**

(iii) Find the value of the machine, and the time that has elapsed, when the machine is losing value at a rate equal to one quarter of the initial rate at which it loses value. **2**

Proceed to next page for question (*14*)

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Question 14 (15 marks) Start a NEW page

(a)! A particle is moving with simple harmonic motion in a straight line.
It has amplitude of *10* metres and a period of *10* seconds.

(i) Prove that it would take the particle $\frac{5}{\pi}\cos^{-1}\frac{3}{5}$ sec to travel from one of the extremities of its path to a point *4* metres away? **2**

(ii) At what speed, correct to whole m/s, would the particle reach this position? **1**

(b)! It is known that $\ln x + \sin x = 0$ has a root close to $x = 0{\cdot}5$. Use one application of Newton's Method to obtain a better approximation of the root to 4 decimal places. **2**

(c)! A projectile with initial velocity U ms^{-1} at an angle of projection α, and acceleration downwards due to gravity, g, has been fired from the origin.

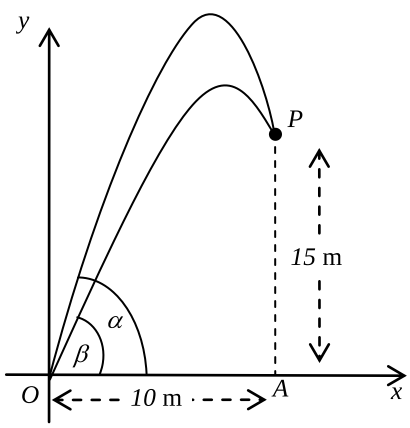
(i) At a time $t \geq 0$ seconds the projectile is at the point (x, y), prove that **3**

$$x = Ut\cos\alpha \qquad \text{and} \qquad y = Ut\sin\alpha - \frac{1}{2}gt^2$$

(ii) Show that the equation of the path of a projectile is given by **2**

$$y = x\tan\alpha - \frac{gx^2}{2U^2}\sec^2\alpha$$

Nicholas throws a small pebble from a fixed point O on level ground, with a velocity $U = 7\sqrt{10}\ ms^{-1}$ at an angle α, with the horizontal. Shortly afterwards, he throws another small pebble from the same point at the same speed but at a different angle to the horizontal β, where $\beta < \alpha$, as shown. The pebbles collided at a point $P(10, 15)$.
Consider the acceleration downwards due to gravity is $g = 9{\cdot}8$ ms^{-1}.



(iii) Show that the two possible initial angles of projection are $\alpha = \tan^{-1}8$ and $\beta = \tan^{-1}2$ **3**

(iv) Show that the time elapsed between when the pebbles were thrown was $\frac{\sqrt{650}-\sqrt{50}}{7}$ seconds. **2**

End of paper

[[End Of Qns]]

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a \neq 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x > 0$

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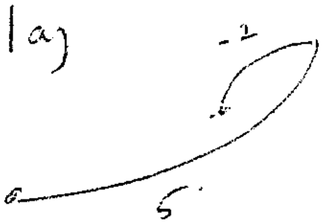
HSC Trial Ext(1)
2013
Mult. Choice Answers

Section (I)

1)	C	6)	D
2)	B	7)	D
3)	B	8)	A
4)	D	9)	A
5)	A	10)	C

11 a)



$\frac{5\times1+-2\times-5}{3}, \frac{5\times3+-2\times6}{3}$

$= \frac{15}{3}, \frac{3}{3}$

$=(5,1)$

b) $x-y+3=0$

perp distance

$\left|\frac{-1x_1-1y_1+3}{\sqrt{2}}\right| = 2\sqrt{2}$.

$x-y+3=\pm4$.

$y=3 \rightarrow$ from distance to x axis is 3

~~x-y=1~~

when $x=4$ $\quad y=\pm3$

when $y=-3$ $\quad x=-10$.

or $x=-2$

$y=3$ $\quad x=-4$

or $x=4$

$(\pm4,3), (-2,-3), (-10,-3)$

(c) $(2x^2-1)^2 = \frac{(2x^2-1)^2}{(2-4x^2)}$

$(2x^2-1)^2(2-4x^2)-(2x^2-1)^2=0$

$(2x^2-1)^2[2-4x^2-1]=0$

$(2x^2-1)^2[1-4x^2]=0$

$(2x^2-1)=0$ or $(1-4x^2)=0$

$x^2=\frac{1}{2}$ $\quad (1-2x)(1+2x)=0$

$x=\pm\frac{1}{\sqrt{2}}$ $\quad x=\pm\frac{1}{2}$.

d) $2x(\sin x)^{\frac{1}{2}}$ $\quad \frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$

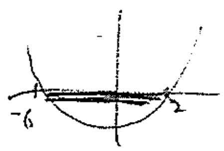
$(\sin x)^{\frac{1}{2}}\cdot 2+2x\cdot\frac{1}{2}(\sin x)^{-\frac{1}{2}}\cdot\cos x$

$=2\sqrt{\sin x}+\frac{x\cos x}{\sqrt{\sin x}}$

$=\frac{2\sin x+x\cos x}{\sqrt{\sin x}}$

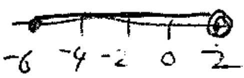
e)(i) $x^2+4x-12\le0$

$(x+6)(x-2)\le0$



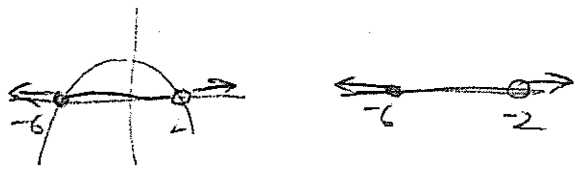
$\therefore -6\le x\le 2$ undefined at $x=2$

$\therefore -6\le x<2$



(ii) $\frac{x+6}{2-x}\le0$

$(x+6)(2-x)\le0$



$\therefore$ $x\le-6$ or $x>2$.

(iii) Simultaneous $x=-6$

(iv) $f(x)=\frac{x^3}{x}$

$g(x)=-\frac{x^3}{x}$

12)

a) $A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$\frac{dr}{dt} = 0.01$

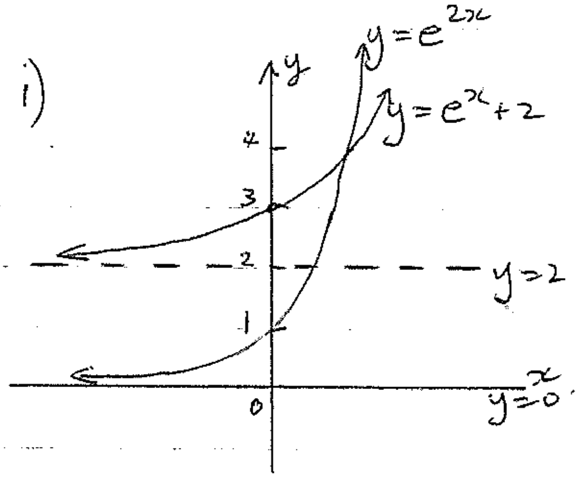
$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$\underline{r=20}$

$\frac{dA}{dr} = 2\pi \times 20 \times 0.01$

$= 0.4\pi$ cm²/sec.

b) i)



ii) $e^{2x} = e^x + 2$

$e^{2x} - e^x - 2 = 0$

$(e^x)^2 - e^x - 2 = 0$

$(e^x - 2)(e^x + 1) = 0$

$\therefore e^x = 2 \quad + \quad e^x = -1$

$x = \ln 2 \qquad$ No. sol^n.

$y = e^{2\ln 2} = e^{\ln 2^2}$

$= e^{\ln 4}$

$= 4 \qquad \therefore (\ln 2, 4)$

iii) $A = \int_0^{\ln 2} \left[e^x + 2 - e^{2x} \right] dx$

$= \left[e^x + 2x - \frac{e^{2x}}{2} \right]_0^{\ln 2}$

$= \left[e^{\ln 2} + 2\ln 2 - \frac{e^{2\ln 2}}{2} \right] - \left[e^0 + 0 - \frac{e^0}{2} \right]$

$= \left(2 + 2\ln 2 - \frac{4}{2}\right) - \left(1 - \frac{1}{2}\right)$

$= 2\ln 2 - \frac{1}{2}$

$\doteqdot 0.89$ (2 sig. fig)

c) $\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$

$y = \ln x \quad \therefore \frac{dy}{dx} = \frac{1}{x}$

for $x = 1$, $m_1 = 1$

for $x = x_1$, $m_2 = \frac{1}{x_1}$

$\therefore \tan\left(\frac{\pi}{6}\right) = \left| \frac{1 - \frac{1}{x_1}}{1 + 1 \cdot \frac{1}{x_1}} \right|$

$\frac{1}{\sqrt{3}} = \left| \frac{\frac{x_1 - 1}{x_1}}{\frac{x_1 + 1}{x_1}} \right|$

$\frac{1}{\sqrt{3}} = \left| \frac{x_1 - 1}{x_1 + 1} \right|$

$\therefore \frac{x_1 - 1}{x_1 + 1} = \frac{1}{\sqrt{3}}$ or $\frac{x_1 - 1}{x_1 + 1} = \frac{-1}{\sqrt{3}}$

$\sqrt{3}x_1 - \sqrt{3} = x_1 + 1 \qquad \sqrt{3}x_1 - \sqrt{3} = -x_1 - 1$

$\sqrt{3}x_1 - x_1 = \sqrt{3} + 1 \qquad \sqrt{3}x_1 + x_1 = \sqrt{3} - 1$

$x_1 = \frac{\sqrt{3}+1}{\sqrt{3}-1} \qquad x_1 = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

$= 2 + \sqrt{3} \qquad = 2 - \sqrt{3}$

12) d) i)

$y = \sin^{-1}(1-x^2)$ Let $u = 1-x^2$

$y = \sin^{-1}(u)$ $\quad \frac{du}{dx} = -2x$

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$= \frac{1}{\sqrt{1-(1-x^2)^2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

$$= \frac{1}{\sqrt{1-(1-2x^2+x^4)}} \times -2x$$

$$= \frac{-2x}{\sqrt{1-1+2x^2-x^4}}$$

$$= \frac{-2x}{\sqrt{x^2(2-x^2)}}$$

$$= \frac{-2x}{x\sqrt{2-x^2}}$$

$$= \frac{-2}{\sqrt{2-x^2}}$$

ii) $f(x) = 2\cos^{-1}\left(\frac{x}{\sqrt{2}}\right) - \sin^{-1}(1-x^2)$

$$f'(x) = 2 \cdot \frac{-1}{\sqrt{(\sqrt{2})^2 - x^2}} - \frac{-2}{\sqrt{2-x^2}}$$

$$= \frac{-2}{\sqrt{2-x^2}} + \frac{2}{\sqrt{2-x^2}}$$

$$= 0$$

iii) If $f'(x) = 0$ then $y = f(x)$ is a horizontal line.

i.e. $y = c$

sub any x value to find c.

$$f(0) = 2\cos^{-1}\left(\frac{0}{\sqrt{2}}\right) - \sin^{-1}(1-0^2)$$

$$= 2\cos^{-1}(0) - \sin^{-1}(1)$$

$$= 2 \times \frac{\pi}{2} - \frac{\pi}{2}$$

$$= \frac{\pi}{2}$$

$$\therefore f(x) = \frac{\pi}{2}$$

Q13 Ext I 2013 5

a) $\tan^{-1}(-\sqrt{3})$

S | A
T | C

0 to 2π.

$= \pi - \frac{\pi}{3}$ or $2\pi - \frac{\pi}{3} = \boxed{\frac{2\pi}{3}}$ or $\boxed{\frac{5\pi}{3}}$

b) $\int_0^1 \sqrt{1-x^2}\,dx \qquad x = \sin\theta$

$= \int \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta \qquad \frac{dx}{d\theta} = \cos\theta$

$dx = \cos\theta\,d\theta$

$= \int \sqrt{\cos^2\theta}\,\cos\theta\,d\theta$

$= \int \cos\theta \cdot \cos\theta\,d\theta = \int \cos^2\theta\,d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + \cos 2\theta\,d\theta$

As
$x = 0, \quad \theta = 0$
$\cancel{\sin}\,x = 1, \quad \theta = \frac{\pi}{2}$

$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left(\left(\frac{\pi}{2} + \underbrace{\frac{\sin\pi}{2}}_{=0}\right) - 0\right)$

$= \frac{1}{2} \times \frac{\pi}{2} = \boxed{\frac{\pi}{4}}$

c) i) $ax^2 + bx + c$ neg def

$\underline{a < 0, \ \Delta = b^2 - 4ac < 0}$

(ii) $a = k^2 + k \qquad b = 6 - 2k \qquad c = 2$

$\Delta = b^2 - 4ac = (6-2k)^2 - 4 \times (k^2+k) \times 2$

$= 36 - 24k + 4k^2 - 8k^2 - 8k$

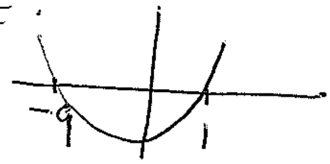
$= -4k^2 - 32k + 36$

$= -4(k^2 + 8k - 9)$

$= -4(k+9)(k-1)$ Solve < 0

$\therefore$ Solve $(k+9)(k-1) > 0$

c(ii) cont.

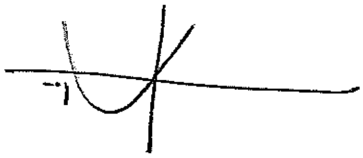


$k > 1$ or $k < -9$ (A)

6

Also $a = k^2 + k < 0$

$k(k+1) < 0$



$-1 < k < 0$ (B)

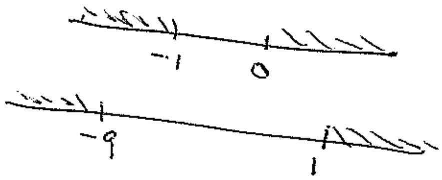
Both (A) and (B) must hold – no simultaneous solution. ∴ cannot be negative definite.

(iii) +ve def $a > 0$, $\Delta < 0$

$k(k+1) > 0$ $\begin{cases} k > 0 \text{ or } k < -1 \\ k > 1 \text{ or } k < -9 \text{ (from above)} \end{cases}$

$\Delta < 0$

Solve simultaneously



∴ $k > 1$ or $k < -9$

d) $2 \times 1! + 5 \times 2! + 10 \times 3! + \ldots + (n^2+1) \times n! = n(n+1)!$ (1)

Step 1 Show statement true for $n = 1$

LHS of (1) $= (1^2+1) \times 1! = 2$

RHS of (1) $= 1(1+1)! = 2$

LHS = RHS

∴ Statement true for $n = 1$

Step 2 Assume statement true for $n = k$, ie

assume $2 \times 1! + 5 \times 2! + \ldots + (k^2+1) \times k! = k(k+1)!$

Prove statement is true for $n = k+1$, ie prove

$2 \times 1! + 5 \times 2! + \ldots + (k^2+1) \times k! + ((k+1)^2+1) \times (k+1)!$

$= (k+1)((k+1)+1)!$ (2)

Proof: LHS of (2) $= k(k+1)! + (k^2+2k+1+1)(k+1)!$

By assumption

$= (k+1)!\,(k + k^2 + 2k + 2)$

$= (k+1)!\,(k^2 + 3k + 2)$

$= (k+1)!\,(k+2)(k+1) = (k+2)!\,(k+1) = (k+1)(k+1+1)!$

= RHS of (2)

d) cont. $\therefore$ Statement true for $n = k+1$ (7)

<u>Step 3</u> Hence, by mathematical induction statement true for all positive integers $n \geq 1$.

e) $\angle X = 35°$ (given)
$\angle PQR = 25°$ (given)
$\angle PRQ = 35° = \angle X$ (angles in same segment are equal)
$\therefore \angle RPQ = 120°$ ($\angle$ sum triangle $RPQ = 180°$)
$\angle XPY = 90°$ ($\angle$ in semi circle is right angle)
$\angle YPR = 90°$ (adjacent, supplementary to $\angle XPY$)
$\angle YPR = \angle RPQ - \angle YPQ$
$= 120° - 90°$
$= \underline{\underline{30°}}$

3U - Trial 2013
Qn(14)

8

a $a = 10m$, $P = 10$ sec

$n = \frac{2\pi}{P} = \frac{\pi}{5}$ rad/sec. $(\frac{1}{2})$

(i) $x = 10\cos(nt + \alpha)$

$t = 0$, $x = 10$ m $\Longrightarrow$ $(\frac{1}{2})$

$10 = 10\cos\alpha \Rightarrow \alpha = 0$

$\Rightarrow x = 10\cos\frac{\pi}{5}t$

$x = 6$ m $(\frac{1}{2})$

$\Rightarrow 6 = 10\cos\frac{\pi}{5}t$

$\Rightarrow t = \frac{5}{\pi}\cos^{-1}\frac{3}{5}$ $(\frac{1}{2})$

/2

b $f(x) = \ln x + \sin x$

$(\frac{1}{2})$

~~let~~ $f'(x) = \frac{1}{x} + \cos x$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$(\frac{1}{2})$ $(\frac{1}{2})$

$= 0.5 - \frac{\ln(0.5) + \sin(0.5)}{2 + \cos(0.5)}$

$= 0.5 - \frac{-0.2137216}{2.87758}$

$= 0.5742712$

~~≈ 0.5781~~ --- $(\frac{1}{2})$

≈ 0.5743

/2

(ii) $\dot{x} = -2\pi\sin\frac{\pi}{5}t$ $(\frac{1}{2})$

The speed $= |\dot{x}|$

$= 2\pi\sin\frac{\pi}{5}\cdot\frac{5}{\pi}\cos^{-1}\frac{3}{5}$

$= 2\pi \times \frac{4}{5}$

$= \frac{8\pi}{5} \approx 5$ m/s $(\frac{1}{2})$

no penalty for ± 5 m/s

/1

OR

$V^2 = n^2(a^2 - x^2)$

$= \frac{\pi^2}{25}(100 - 36)$

$= \frac{64\pi^2}{25}$

$V = \frac{8\pi}{5} \approx 5$ m/s

no penalty for ± 5 m/s

Qn (14): Continued

$\boxed{C}$ (i) $\ddot{x} = 0$ —— ($\frac{1}{2}$)

$\dot{x} = C$ or

$\dot{x} = U\cos\alpha$ or

$x = Ut\cos\alpha + C$

$t = 0,\ x = 0 \Rightarrow C = 0$

$\Rightarrow \underline{x = Ut\cos\alpha}$ —— ($\frac{1}{2}$)

$\ddot{y} = -g$ —— ($\frac{1}{2}$)

$\dot{y} = -gt + C$

$t = 0,\ \dot{y} = U\sin\alpha$

$\Rightarrow C = U\sin\alpha$

$\Rightarrow \dot{y} = U\sin\alpha - gt$ —— ($\frac{1}{2}$)

$y = Ut\sin\alpha - \frac{1}{2}gt^2 + C$

$t = 0,\ y = 0 \Rightarrow C = 0$ ($\frac{1}{2}$)

$\Rightarrow \underline{y = Ut\sin\alpha - \frac{1}{2}gt^2}$ ($\frac{1}{2}$)

/3

(ii) $t = \dfrac{x}{U\cos\alpha}$ —— ($\frac{1}{2}$)

sub. into y,

($\frac{1}{2}$)

$y = (\cancel{U}\sin\alpha)\dfrac{x}{\cancel{U}\cos\alpha} - \frac{1}{2}g\left(\dfrac{x}{U\cos\alpha}\right)^2$

$= x\tan\alpha - \dfrac{gx^2}{2U^2\cos^2\alpha}$

$y = x\tan\alpha - \dfrac{gx^2}{2U^2}\sec^2\alpha$ (1)

/2

Qn(14): Continued

C-iii) $U = 7\sqrt{10}$ m/s

$x = 10$ m, $y = 15$ m

Substitute into the eqn of the path:

$$y = 10\tan\alpha - \frac{9.8 \times 100}{2 \times 490}\sec^2\alpha$$

$$\tan^2\alpha - 10\tan\alpha + 16 = 0$$

$$\tan\alpha = \frac{10 \pm \sqrt{100 - 4 \times 1 \times 16}}{2}$$

$$= 5 \pm 3$$

$\tan\alpha = 8$ or $\tan\beta = 2$

C-iv) Let t_1 & t_2 be the projection time of the 2 pebbles.

$$t_1 = \frac{x}{U\cos\alpha}$$

$$t_2 = \frac{x}{U\cos\beta}$$

Time elapsed,

$$t_1 - t_2 = \frac{x}{U\cos\alpha} - \frac{x}{U\cos\beta}$$

$$= \frac{x}{U}(\sec\alpha - \sec\beta)$$

$$= \frac{10}{7\sqrt{10}}(\sqrt{65} - \sqrt{5})$$

$$= \frac{\sqrt{650} - \sqrt{50}}{7}$$